Technical Note: Structural Decomposition of HD±DH for Walsh–Hadamard Mixing Systems

---

1. Overview

Consider:

$$H(D \pm HDH)$$

where:

- H is the normalized Walsh–Hadamard transform matrix
- D is diagonal, often with entries:

  $d_i \in \{+1, -1\}$

Using:

$$H^2 = I$$

this simplifies to:

$$HD \pm DH$$

This decomposition reveals two fundamentally different interaction mechanisms:

- **HD + DH** → agreement / reinforcement operator
- **HD - DH** → disagreement / interaction operator

The structure becomes especially transparent when D is binary ±1.

Part I: Intuitive Structural View

---

2. Grounding: What the operators mean

D: local channel labeling

D acts independently per coordinate.

For binary D:

- +1: one class / orientation / polarity
- -1: opposite class

Thus D assigns every channel a binary label.

H: global mixing

The Walsh–Hadamard transform mixes all coordinates through equal-magnitude ± interactions.

Each output coordinate depends on every input coordinate.

HD vs DH

These correspond to two different operation orderings:

HD

scale first, then mix

DH

mix first, then scale

Their comparison reveals whether local labeling and global mixing are compatible.

---

3. Entrywise structure

For binary diagonal:

$(HD + DH)_{ij} = H_{ij}(d_i + d_j)$

$(HD - DH)_{ij} = H_{ij}(d_j - d_i)$

Since:

$d_i, d_j \in \{+1, -1\}$

only three values occur:

+2, 0, -2

This gives a simple interaction logic.

---

4. Agreement structure of HD+DH

Positive agreement

If:

$d_i = d_j = +1$

then:

$d_i+d_j = +2$

Entry survives positively.

Interpretation:

- same orientation
- mutually reinforcing

Negative agreement

If:

$d_i=d_j = -1$

then:

$d_i + d_j = -2$

Entry survives with opposite sign.

Interpretation:

- still agreement
- but in opposite polarity

Disagreement

If:

$d_i \neq d_j$

then:

$d_i + d_j = 0$

Entry cancels.

Interpretation:

- incompatible channels do not interact

---

5. Meaning of (HD+DH)

This operator performs:

Agreement detection

Only channels with matching labels interact.

Reinforcement

Matching interactions are doubled.

Cluster preservation

Channels self-organize into internally coherent groups.

A useful mental model:

"Only like talks to like."

---

6. Structure of (HD-DH)

Now:

$(HD\text{-}DH)_{ij} = H_{ij}(d_j - d_i)$

Agreement

If:

$d_i = d_j$

Then:

0

Agreement cancels.

Disagreement

If:

$d_i \neq d_j$

interaction survives.

---

7. Directed disagreement

This is the key distinction.

For opposite labels:

+1 → -1

one sign

-1 → +1

opposite sign

Thus disagreement is not merely detected.

It is **oriented**.

---

8. Meaning of HD-DH

This operator performs:

Disagreement detection

Only opposite-label channels interact.

Direction assignment

Interaction carries orientation.

Cross-group transport

Information moves between opposing groups.

Mental model:

"Only unlike channels talk, and the sign tells you who pushes whom."

---

9. Cancellation / reinforcement summary

HD+DH

relation	result
positive agreement	+2
disagreement	0
negative agreement	-2

Agreement survives.

HD-DH

relation	result
agreement	0
positive disagreement	+2
negative disagreement	-2

Disagreement survives.

---

10. Entropy interpretation

Let:

$P(d_i = +1) = p$

Then:

Agreement probability

$p^2 + (1-p)^2$

Disagreement probability

$2p(1 - p)$

maximized at:

$p=0.5$

Implication:

- uniform labeling → no disagreement interaction
- balanced labeling → maximal disagreement interaction

Thus:

interaction strength tracks partition entropy

---

11. Practical reasoning rule

To reason about the system:

Ask:

Are two channels labeled the same?

If yes:

- contributes to HD+DH

If no:

- contributes to HD-DH

This is often enough to reason about behavior without full matrix algebra.

Part II: Higher-Math Research Directions

---

12. Operator-theoretic structure

Anti-commutator

$$\{H,D\}=HD+DH$$

Relevant to:

- Jordan algebra structure
- symmetrized operator products
- cooperative basis interactions

Investigate:

- spectral clustering under binary diagonal perturbations
- eigenspace localization

Commutator

$$[H,D]=HD-DH$$

Measures non-commutativity.

Investigate:

- Lie algebra generation

- BCH expansions
- discrete bracket dynamics

---

13. Discrete Lie-flow interpretation

Alternating application of:

- local scaling
- global Hadamard mixing

produces residual drift represented by:

[H,D]

Interpret as:

- discrete transport generator
- finite-dimensional bracket flow analogue

Potential connection to:

- operator splitting
- symplectic discretization
- Trotter product expansions

---

14. Second commutator / diffusion

[H,[H,D]]

2(D-HDH)

This is a discrete diffusion / smoothing operator.

Investigate:

- Laplacian analogues
- entropy dissipation
- contractive semigroups

---

15. Signed graph interpretation

Binary D defines a partition.

Then:

HD+DH

intra-cluster adjacency

HD-DH

inter-cluster directed cut operator

Investigate:

- signed graph spectra
- cut energies
- balance theory
- bipartite flow systems

---

16. Incidence / Laplacian factorization

Potential factorization:

$$D\text{-}HDH \sim BWB^T$$

where:

- B: dense signed incidence operator
- W: Walsh-derived weighting

Research direction:

explicit construction.

This would formalize:

- transport
- divergence
- diffusion

within the Walsh basis.

---

17. Entropy-linked interaction scaling

Expected interaction strength:

$$\sim \sqrt{n\,p(1-p)}$$

Investigate:

- statistical mechanics interpretations
- entropy-driven operator phase transitions
- random binary diagonal ensembles

---

18. Spectral questions

Key open directions:

Eigenvalue distribution of

HD-DH

for random binary D

Spectral gap of

D-HDH

Localization / delocalization transitions

as entropy varies.

Potential connections to:

- random matrix theory
- free probability
- spin-glass-like ensembles

---

19. Neural / computational interpretations

Possible uses:

HD+DH

structure-preserving mixing

HD-DH

cross-channel transport / contrastive exchange

Relevant to:

- reversible architectures
- information-preserving transforms
- structured orthogonal mixing layers

---

20. Core conceptual takeaway

The binary Walsh–diagonal system naturally decomposes into:

Agreement geometry

HD+DH

Disagreement transport

HD-DH

Diffusive equalization

D-HDH

This gives a compact algebraic framework for:

- clustering
- transport
- entropy
- diffusion
- noncommutative dynamics

all emerging from the interaction of a global orthogonal mixer with local binary structure.